Solitons and wavelets: Scale analysis and bases

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Abstract

We use a one-scale similarity analysis which gives specific relations between the velocity, amplitude and width of localized solutions of nonlinear differential equations, whose exact solutions are generally difficult to obtain. We also introduce kink-antikink compact solutions for the nonlinear-nonlinear dispersion K(2,2) equation, and we construct a basis of scaling functions similar with those used in the multiresolution analysis. These approaches are useful in describing nonlinear structures and patterns, as well as in the derivation of the time evolution of initial data for nonlinear equations with finite wavelength soliton solutions.

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1 Introduction

The nonlinear partial differential equations (NPDE) of physical interest can describe a variety of patterns, particle-like traveling solutions, solitons and breather modes in nuclear and particle physics, nonlinear molecular and solid state physics phenomena, and features found in nonlinear optics [1]. Their solutions are usually localized and demonstrate stability in time and in collisions with each other. In the asymptotic domain these solutions consist of isolated traveling pulses that are free of interactions and have a shape related to the velocity, thus making nonlinear patterns distinct from linear results. In the scattering domain, the nonlinear solutions obey some nonlinear superposition principle.

Nonlinear dynamics require NPDE which display very strong interaction between the initial conditions and the dynamics and involve multiple scales [2, 3], being able to produce self-similar or fractal-like patterns. Recent examples show that the traditional nonlinear tools (inverse scattering, group symmetry, functional transforms) are not always applicable [4]. From the experimental point of view one knows that such patterns generally have finite space-time extension and a multi-scale structure. Since the traditional solitons or the soliton-like solutions have infinite extent, one needs rather appropriate compact supported basis functions to investigate such structures.

Multi-resolution analysis (MRA), [5], could be a useful method for the construction of such nonlinear bases, since the linear harmonic analysis is inadequate for describing nonlinear systems. The MRA uses wavelets, which are functions that have a space-dependent scale which renders them an invaluable tool for analyzing multi-scale phenomena. Wavelets have been used in signal processing, problems involving singular potentials in quantum mechanics, in discussions concerning q-algebras, and even in nuclear structure studies [6]. It follows that the use of MRA in the study of NPDE is natural because wavelets can analyse nonlinear features like strong variations or singularities. The NPDE describe a variety of patterns [2], features in quantum optics [7], molecular and solid state physics phenomena [8] and solitons in nuclear and particle physics [9, 10].

In this paper, wavelet-inspired approaches for localized solutions of NPDE are explored. We propose two different formalisms for the scale analysis, and the classification of soliton solutions of NPDE. A first method provides relations between the characteristics of such solutions (amplitude, width and velocity) without the need of solving the corresponding NPDE. The method uses the multi-resolution analysis [5] instead of the traditional tools like the Fourier integrals or linear harmonic analysis which are inadequate for describing such systems. Moreover, the introduction of wavelet analysis in the study of NPDE is somehow natural because it can accommodate behaviour diplaying strong variations, even singularities, to a smooth behavior. This scale approach has the advantage that it does not need the explicit form of the exact solutions. Hence, it is useful especially in situations when such solutions are unknown. In section 2 we provide many examples, predictions and applications of this one-scale approximation method (OSA) for a large class of NPDE with respect to their localized solutions. The NPDE is mapped into an algebraic equation relating the amplitude, width and velocity of such signals. The results are succinctly presented in tabular form. In section 2 we show an example of the construction of a nonlinear basis for NPDE i.e. the construction of a kink-antikink basis for the K(2,2) equation, involving nonlinear dispersion. There are many physical reasons

favoring wavelets in the construction of such nonlinear bases. The self-similar character of the fission process of fluid drops is an example where the same type of singularity occurs in any scale [2, 11, 12]. In section 3, we use also this basis (or frame) for the investigation of the time evolution of a given initial data profile for the nonlinear-nonlinear dispersion K(2,2) equation. The proofs are introduced in two Appendices.

2 Scale analysis of NPDE

In this section we introduce a one-scale analysis (OSA) for the NPDE, in terms of their localized traveling solutions. The NPDE which describe physical phenomena are usually not susceptible to analytic solutions. Moreover, there are examples [4, 13] when the mathematical tools like the inverse scattering theory or the transformation group method are not applicable. When phenomena of interest have many space-time scales, or the scale of the process varies in time, the numerical methods may fail, like in the case of propagating discontinuities or shock waves. A simple option is the expansion of solutions in a basis of appropriate chosen basis. The Fourier series, which has the advantage of orthogonality, cannot discriminate the local behavior of phenomena. Moreover, the analysis of the Fourier coefficients of a wave $u(x,t)$ is not sufficient for drawing conclusions about the scale of the localized structures.

The so called OSA analysis, described and applied here for localized traveling solutions belonging to any type of NPDE, provides algebraic connections between the width L, amplitude A, and the velocity V of the solution, without actually solving the equation. The procedure consists in the substitution of all the terms in the NPDE, according to the rules:

$$\begin{aligned} &\text{i.} \quad u_t \to -V u_x \quad \text{substitution for traveling solutions,} \\ &\text{ii.} \quad u \to \pm A, \qquad u_x \to \pm A/L, \qquad u_{xx} \to \pm A/L^2 \ldots, \end{aligned} \tag{1}$$

and so forth for higher order of derivatives. This substitution in eq.(1 ii) is possible only for localized (finite extended support) solutions having at least one local maximum (like solitons or Gauss functions), if they exist. The advantage of the substitution follows from the fact that it enables one to say something about the hypothetical localized traveling solutions even if (or especially when) they are not known or discovered. However, this averaging method may fail in the case of strictly monotonic solutions, or solutions with singularities. This will be made more explicit in the next paragraph and Appendix 1.

Since we are interested in traveling solutions, the first substitution reduces the number of variables from 2 to 1 so that we are now dealing with an ordinary differential equation instead of a PDE. Then, the second substitution transforms the ordinary differential equation into an equation in the parameters describing the amplitude, width and velocity. Consequently, the NPDE is mapped into an algebraic equation in A, L and V.

The proof of the method follows from the expansion of the soliton-like solution $u(x,t) = u(s)$ with $s = x - Vt$, in a Gaussian family of wavelets $\Psi(s) = Ne^{Q(s)}$, where $Q(s)$ is a polynomial and N the normalization constant [5, 14]. If we choose $Q = -is - \frac{s^2}{2}$ we obtain a very particular wavelet with the support mainly confined in the $(-1,1)$ interval, namely $\Psi(s) = \exp[-is - \frac{s^2}{2}]/\pi^{1/4}$. We have the discrete wavelet expansion of u

$$u(s) = \sum_j \sum_k C_{j,k} \Psi(2^j s - k) = \sum_{j,k} C_{j,k} \Psi_{j,k}(s), \qquad (2)$$

in terms of integer translations (k) of Ψ, which provide the analysis of localization, and in terms if dyadic dilations (2^j) of Ψ, which provide the description of different scales. The idea of the proof is to choose a point where the expansion in eq.(2) can be well approximated by one single scale such that $u(s)$ can be approximated with a sum of phases in s. In order to reduce the number of scales, the range of the summation should be chosen with an eye to the underlying physics. The exact proof of this approach is provided in Appendix 1.

In the following we apply this procedure to three classes of NPDE, namely: dispersive, diffusive and dispersive-diffusive nonlinear equations, and we present the results in tabular form. Before describing different examples of NPDE, we point out that for each equation we try to present one example of an exact solution (second column Tables 1,2) in comparison with the results of the much simplest OSA approach (third column of first two tables and second column of Table 3). We stress that we do not choose only that solution which fits the similarity analysis. On the contrary, our approach fits a large number of solutions, as can be checked directly.

Our first example of application of the OSA approach is given by the convective-dispersive equations, for which the most celebrated example is provided by the KdV equation

$$u_t + uu_x + u_{xxx} = 0. \qquad (3)$$

In general, the stable solutions of the nonlinear-dispersive equations are dependent of the initial conditions, through their conservation laws. Consequently, they can gener-

ate a large class of patterns, shaped by the balance between nonlinear interaction and dispersion, among which the most interesting examples are solitons, breathers and kinks.

In Table 1 we present examples of pure dispersive NPDE, identified in the first column by the form of the equation, and in the second column by a corresponding traveling localized solution, if the analytical form is available. Such exact solutions provide special relations between L, A and V, which are given in the third column of Table 1. In the last column we introduce the results of OSA, namely the relations between these three parameters, provided by eqs.(1 ii). The usefulness of the approach may be checked, by a quick comparison between the second and the third columns. While the results in the second column are possible only when one knows the analytical solutions, the results presented in the last column, obtained by the OSA, result directly from the NPDE form, without actually solving it.

The case of the KdV equation, eq(3), is described in the first row of the Table 1. The OSA method gives a general expression for $L = L(A, V)$, shown in the first line, first row of the last column. In the second line of the last column we show that a specification of the relation between V and A, together with the general result given in the first line, results in a relation between L and A. For L to be related to A only, it results from the third column that the velocity V must be proportional to A. In this case we obtain the well-known relation (first row, column two) among the parameters in the soliton solution $V = 2A$. Moreover, OSA method allows V to depend on a higher power of A ($V \sim A^p$, $p \geq 1$). If such a solution could exist, a lower bound for A will occur. Such solitons would have only amplitudes higher than this limit, while solitons with a smaller amplitude than this limit move with velocity proportional to A.

Similar results are obtained for the MKdV equation (second row), except that here A needs to be proportional to the square root of V in order to have L a function of A only. This prediction is again identical with that in the exact solution (second column). Moreover, the same relations remain valid even for the solutions of the "compacton" type [15]

$$u(x,t) = \frac{\sqrt{32}k\cos[k(x-4k^2t)]^2}{3(1-\frac{2}{3}\cos[k(x-4k^2t)]^2)},$$

where $L = \pi/6k$, that is $L \sim 1/A$, like in the Table 1.

Next example (third row) is provided by a generalised KdV equation, in which the dispersion term is quadratic

$$u_t + (u^2)_x + (u^2)_{xxx} = 0. \tag{4}$$

Eq.(4), known as K(2,2) equation because of the two quadratic terms, admits compact supported traveling solutions, named compactons [4, 13, 16, 17, 18]. In general, the compactons are obtained in the form of a power of some trigonometric function defined only on its half-period, and zero otherwise, in such a way that the solution is enough smooth for the NPDE in discussion. In the above example the square of the solution has to be continuous up to its third derivative with respect to x.

Different from solitons, the compacton width is independent of the amplitude and this fact provides the special connection with the wavelet bases. The compactons are characterize by a unique scale, and it is this feature that makes it possible to introduce a nonlinear basis starting from a unique generic function. For eq.(4) the compacton solution is given by

$$\eta_c(x - Vt) = \frac{4V}{3} cos^2\left[\frac{x - Vt}{4}\right], \tag{5}$$

if $|x - Vt| < 2\pi$ and zero otherwise. Here we notice that the velocity is proportional to the amplitude and the width of the wave is independent of the amplitude, $L = 4$. As a field of application we mention that the quadratic dispersion term is characteristic for the dynamics of a chain with nonlinear coupling.

The general compacton solution for eq.(4) is actually a "dilated" version of eq.(5). That is, a combination of the first rising half of the squared cos in eq.(5), followed by a flat domain of arbitrary length λ, and finally followed by the second, descending part of eq.(5). Actually, this combination is just a kink compacton joined smoothly with an antikink one

$$\eta_{kak}(x - Vt; \lambda) = \begin{cases} 0 \ldots \\ \frac{4V}{3}\ cos^2\left[\frac{x-Vt}{4}\right], \quad -2\pi \le x - Vt \le 0 \\ \frac{4V}{3}, \quad 0 \le x - Vt \le \lambda \\ \frac{4V}{3}\ cos^2\left[\frac{x-Vt-\lambda}{4}\right], \ \lambda \le x - Vt \le \lambda + 2\pi \\ 0 \ldots \end{cases} \tag{6}$$

In Fig. 1 we present compactons (upper line), kink-antikink pairs (KAK) described by eq.(6), both with the same amplitude and velocity (middle line). Although the second derivative of this generalized compacton is discontinuous at its edges, the KAK, eq.(6), is still a solution of eq.(4) because the third derivative acts on u^2, which is a function of class C_3. Finally, we can construct solutions by placing a compacton on the top of a

KAK, as in the bottom line of Fig. 1. Such a solution exists only for a short interval of time (λ/V), since the two structures have different velocities. The analytic expression of the solution is given by

$$\eta(x,t) = \eta_{kak}(x - Vt; \lambda) + \left(\eta_c(x - V't - 2\pi) + \frac{4V}{3}\right)\chi(\frac{x - V't - 2\pi}{2\pi}), \tag{7}$$

for $0 < t < (\lambda - 4\pi)/(V' - V)$ and zero in the rest. Here $\chi(x)$ is the support function, equal with 1 for $|x| \leq 1$ and 0 in the rest, and $V' = 3\max\{\eta_c\}/4 + 2V$.

For the K(2,2) compacton eq.(5) fulfills some relations between the parameters: $A = 4V/3$ and $L = 4$ [16]. The relations provided by OSA in the last column of the third row, predict such relations, and hence also proove the existence of the compacton. That is, for a linear relation between the amplitude and the speed, the half-width is constant and does not depend on A. Indeed, if we choose $V = \pm 3A/2$ or $V = \pm 5A/2$, we obtain $L = 4$, like for the compacton. The constant value of $L \equiv L_0 = const.$ is a typical feature of the K(2,2) compactons. Moreover, it was found numerically that for any compact supported initial data, wider than L_0, the solution decomposes in time into a series of L_0 compactons, Fig. 2. For narrower initial data the numeric solution blows up. There is no exact or analytic explanation of this effect, so far. The scale relations can give a hint in this situation, too, by using the graph of the relation $L = L(V, A)$ provided by this qualitative method. In Fig. 3, L from the third column of the third row is plotted versus V, for several values of A (larger values of A translate the curves to the right). Above the value $L = 4$ of the half-width of a stable compacton, wider compact pulses produce an intersection for each curve (each A) with the axis $L = 4$, providing a series of compactons of different heights, like in the numerical experiments described in [4, 13, 17]. Below the $L = 4$ line, all the curves $L(V)$ approach $L = 0$, towards infinite amplitude, explaining the instability of the narrower initial data.

Another good example of the predictive power of the method is exemplified in the case of a general convection-nonlinear dispersion equations, denoted by K(n,m)

$$\eta_t + (\eta^n)_x + (\eta^m)_{xxx} = 0. \tag{8}$$

Compacton solution for any $n \neq m$ are not known in general, except for some particular cases. In this case we find a general relation among the parameters, for any n, m, shown in the fourth and fifth rows. These general relations $L(A, V)$ approach the known relations for the exact solutions, in the particular cases like $n = m$ (fourth row), $n = m = 2$ (third

row), $n = m = 3$ (first reference in [4]) and $n = 3, m = 2$; $n = 2, m = 3$ (fifth row). These results can be used to predict the behavior of solutions for all values of n, m.

In the following, we present another example of applications of the OSA approach, related to a new type of behavior of nonlinear systems. Traditional solitons move with constant speed on a rectilinear path (except for the roton [11] which has a circular trajectory with constant angular velocity). The speed is usually equal to the amplitude scaled with a constant. Higher solitons travel faster and there are no solitons at rest (zero speed implies zero amplitude). They can travel in both directions with opposite signs for the amplitude. The situation is different in the case of compactons, which allow also stationary solutions. When linear and nonlinear disspersion occur simultaneously, like in the so called K(2,1,2) equation

$$u_t + (u^2)_x + (u)_{xxx} + \epsilon(u^2)_{xxx} = 0,$$

where ϵ is a control parameter, the OSA yields a dependence of the form

$$L = \sqrt{(\pm A + \epsilon)/(V \pm A)},$$

which still provides a constant width if $V = \pm A + 2\epsilon$. In this case, the speed is proportional to the amplitude, but can change its sign even at non-zero amplitude. Solutions with larger amplitude than a critical one ($A_{crit} = \mp 2\epsilon$) move to the right, solutions having the critical amplitude are at rest, and solutions smaller than the critical amplitude move to the left. This behavior was explored in [16], too. However, such a switching of the speed is not necessarily a feature of the nonlinear dispersion. A compacton of amplitude A on the top of a infinite-length KAK solution of amplitude δ

$$u(x,t) = Acos^2\Big(\frac{x - Vt}{4}\Big) + \delta, \tag{9}$$

is still a solution of the K(2,2) equation, with the velocity given by $V = \frac{3}{4}\Big(2\delta + A\Big)$. For $A = -2\delta$ the solution becomes an anti-compacton moving together with the KAK. In the case of a slow-scale time-dependent amplitude the oscillations in amplitude can transform into oscillations in the velocity. The key to such a conversion of oscillations is the coupling between the traditional nonlinear picture (convection-dispersion-diffusion) and the typical Schrödinger terms.

In Table 2 we present another class of NPDE, namely the dissipative ones. These equations generalize the linear wave equation (first row) where there is no typical length

of the traveling solutions. The wavelet analysis provides the correct expression for the dispersion relation ($V = c \rightarrow k^2 = \omega^2/c^2$) with no constraint on either the amplitude A or on the width L.

In the second row we introduce the celebrated Burgers equation which represents the simplest model for the convective-dissipative interaction. Dissipative systems are to a large extent indifferent to how they were initialized, and follow their own intrinsic dynamics. We provide in the second column an analytic solution of the Burgers equation. For some special of values of the integration constants ($2C < V^2$, $D = 0$) the solution becomes a traveling kink

$$u(x,t) = V + \sqrt{V^2 - 2C}\tanh[\sqrt{V^2 - 2C}(x - Vt)]. \tag{10}$$

By applying the OSA approach to the Burger equation (third column) we obtain the same relation between amplitude and half-width, like in the case of the exact solution eq.(10), providing the velocity is proportional with the amplitude.

In the following we apply the OSA approach to investigate a nonlinear Burgers equation

$$u_t + a(u^m)_x - \mu(u^k)_{xx} + cu^\gamma = 0, \tag{11}$$

called quasi-linear parabolic equation [18], and used to describe the flow of fluids in porous media or the transport of thermal energy in plasma. The last term describes the volumetric absorption (Bremsstrahlung for $\gamma = 1/2$, synchrotron radiation for γ in the range $1.5 - 2$, etc). The second term in eq.(11) describes the convection process and the coefficient β ranges from 0 to 1 in the case of difussion in plasma, and further to higher values, for unsaturated porous medium in the presence of gravity. The existence and stability of waves or patterns is strongly dependent on the coefficients a, μ, c, m, k and γ, and at this point the OSA can be useful again since there is no general analytic solution for eq.(11). The result of the OSA approach is presented in the third row of Table 2. The typical scale of patterns depends on the parameters in the equations and the amplitude of the excitations, in a complicated way.

However, in order to test OSA again, we found a simple class of exact solutions when $c = 0$, presented in the fourt row in Table 2, and expressed as the inverse of a degenerated hypergeometric function. In this expression we have $\mathcal{A} = \mu k V^{\alpha-1}((k-1)a^\alpha)^{-1}$, $z = (a/V)u^{m-1}$ and $\alpha = (k-1)/(m-1)$. The asymptotic behavior of the left hand side of the solution given in fourth row, second column, is described by

$$\Gamma(\alpha+1)\left[(-1)^\alpha + \frac{1}{\Gamma(\alpha)}z^{\alpha-1}e^\alpha\right] + \mathcal{O}(1/z).$$

If z approaches $+\infty$ the solution increases indefinitely like an exponential. For $\alpha > 1$ (strong difussion effects), for even k and for even m, the traveling wave $u(x - Vt)$ has a negative singularity towards $-\infty$ at $x + x_0 = \Gamma(\alpha + 1)(-1)^{\alpha} < 0$. For k odd there is also a singularity at $x + x_0 > 0$. These solutions are not likely to provide viable physical results. If k is even and m is odd (the singularity is pushed towards imaginary x), or if $0 < \alpha < 1$, the singularity is eliminated and the solution becomes semi-bounded, like in the particular situations investigated in the article [18]. In this case, OSA provides again the correct relations, since we obtaine the special behavior of the solution if the velocity is proportional to the power $m - 1$ of the amplitude A. Also, we predict the space scale of these semi-compact pulses, namely the length $L = \frac{\mu k^2 A^{k-1}}{V \pm amA^{m-1}}$.

The OSA analysis can be applied in the case of sine-Gordon equation, fifth row of Table 2. The solutions with the velocity proportional with L^2 are characterized through the OSA approach by a transcendental equation in A, identical with the equation fulfilled by the amplitude A of the exact sine-Gordon soliton.

In the sixth row, we present the cubic nonlinear Schrödinger equation (NLS3) which has a soliton solution. This type of equation is applied in nonlinear optics, elementary particle physics or in the polaron model in solid state physics [7, 8, 9]. Recently, different effects of cluster physics could be explained by using the NLS3 equation. In the sixth row of Table 2 we present the NLS3 equation together with its one soliton solution of amplitude η_0, obtained by the inverse scattering method. In the last column we also show the relation between the parameters of a localized solution, obtained by OSA. The equation for $L(A, V)$ is more general than that one fulfilled by the soliton, and hence is related to more general localized solutions. By choosing the velocity proportional to the amplitude, we reobtain the $L \sim 1/A = 1/\eta_0 \sim 1/V$ typical relations for the soliton given in the second column. If a more general solution of the NLS3 equation describes, for example, the dynamics of some cluster states, or the dynamics of hard spheres in a hard core potential model

$$-\frac{\hbar^2}{2m}\Psi_{xx} + (E - V)\Psi + a\Psi^3 = 0,$$

then the L parameter gives an estimation for the wavelength of the wavefunctions, or for the correlation length in a Bose model

$$L = \frac{\hbar}{\sqrt{2m(E - V) + aA^2}} \simeq \frac{\hbar}{\sqrt{2m(E - V)}} \quad \text{for small} \quad A.$$

For the general case of a NLS equation of order n (seventh row), where a general analytical

solution is unknown, the method predicts a special $L = L(A, V)$ dependence, shown in the third column and in Fig. 4. Contrary to third order NLS, where the dependence of L with A is monotonous for $V =\sim \pm A$ ($n = 3$ in Fig. 4), at higher orders than 3, the $L(A)$ function has discontinuities in the first derivative. This wiggle of the function (Fig. 4, $n = 4$) holds at a critical width, possibly producing bifurcations in the solutions and scales. As a consequence, initial data close to this width can split into doublet (or even triplet, for higher order NLS) solutions, with different amplitudes. Such phenomena have been put into evidence in several numerical experiments for quintic nonlinear equations [17, 18, 19].

The final example of Table 2 is provided by the Gross-Pitaevski (GP) mean field equation, which is used to describe the dilute Bose condensate [20]. The scalar field (or order parameter) governed by this equation was shown to behave in a particle manner, too, since it can contain topological deffects, namely dark solitons. The space scale L of such solutions is important, for both the theory and experiment, since is related to the trap dimensions and to the scattering length. In the last row of the Table 2 we give one particular solution of a simplified one-dimensional version of the GP equation [21]

$$i\hbar \frac{\partial \Psi(\vec{x}, t)}{\partial t} = \Big(-\frac{\hbar^2 \triangle}{2m} + V_{ext}(\vec{r}) + \frac{4\pi\hbar^2 a}{m} |\Psi(\vec{x}, t)|^2 \Big) \Psi(\vec{x}, t), \tag{12}$$

where a is the s-wave scattering length and V_{ext} is the confining potential. In the solution provided in the table, the half-width of the exact nonstationary solution is $L = 1/\sqrt{v_c^2 - p^2}$, where v_c is the Landau critical velocity, and $p = \dot{q}(t)$ is the momentum associated with the motion of this disturbance. It is easy to check that the OSA provides a good match with this exact solution, and also L fits the correlation length $l_0 = \sqrt{m/4\pi\hbar^2 a}$. We stress that such estimation of the length is also important in nuclear physics where one can explain the fragmentation process as a bosonization in α-particles, inside the nucleus. Such systems are coherent if the wavelength associated with the cluster (the resulting L in the GP equation) is comparable with the distance between the α-clusters.

The more complex the NPDE is, the richer the conclusions of the OSA approach. A good example of such an analysis is related to the convective-dissipative-dispersive NPDE, example provided by the model equation

$$u_t + a(u^m)_x + b(u^k)_{xx} + c(u^n)_{xxx} = 0 \tag{13}$$

Here m, k and n are integers and the corresponding terms are responsible of the nonlinear

interaction (convective term), dissipation and dispersion [18]. The above equation is related to weakly nonlinear phenomena, and it occurs in modeling porous medium, magma, interfacial phenomena in fluids (and hence applications to drop physics), etc. General solutions are difficult to obtain for such a complex equation. We show in the following that most of the essential conclusions for the behavior of its solutions can be obtain, in a simple way, by the OSA. This approach maps this equation into

$$(amA^{m-1} - V)L^2 - \mu k^2 A^{k-1} L + n^3 A^{n-1} = 0, \tag{14}$$

Table 3, first row. The most symmetric case is obtained when either $V = 0$ (stationary patterns) or $V \sim A^{m-1}$. In this situation the condition to have a monotonous dependence of L as a function of A is $2k = m + n$ which yields a scale structure

$$L = A^{k-m}\Big(\frac{\mu k^2 \pm \sqrt{\mu^2 k^4 - 4mn^3(a - V_0)}}{2m(a - V_0)}\Big) \sim A^{k-m}, \tag{15}$$

where we put $V = mV_0 A^{m-1}$. The condition $2k = m+n$ is just the condition obtained in [18] from a scaling approach. This condition assures the universality of the corresponding patterns, and it is the unique case in which L depends on a power of A. In any other situation, this dependence is more complicated and introduces singularities which broke the self-similarity. In the above cited paper, the author finds out the condition for mass invariance at scaling transformations as $m = n + 2 = k + 1$. In our case we just have to request the product AL (which gives a measure of the mass, or volume of the pattern, like in the case of one-dimensional solitons) to be a constant. This gives the condition $k-m+1 = 0$ which, together with the general invariance condition $2k = m+n$, reproduces $m = n + 2 = k + 1$. In this case we have patterns characterized by a width

$$L = \frac{\mu k^2 \pm \sqrt{\mu^2 k^4 - 4mn^3(a - V_0)}}{2mA(a - V_0)} \rightarrow n^3/A\mu k^2.$$

If $a \sim V_0$ the width approaches $n^3/A\mu k^2$. In order to make L independen of A, like in the compacton case, we need $m = k$, which together with the first invariance condition $2k = m + n$, yields $m = n = k$. This is the exceptional case when the dissipative and dispersive processes have the same scaling, resulting form the invariance of the eq.(13) under the group of scales. Finaly, if we choose $L \sim V$ we obtain the condition $k+1 = 2m$ which (together with $2k = m + n$) is the condition for spiral symmetry and occurence of similarity structures [18].

The above comments are not intended to be a complete study of such a complex equation, but the just a proof of how many conclusions one can obtain from the simple equation in A, L and V, eq.(14).

The next example is provided by one of the most generalized KdV equation, which is generated from the Lagrangian [13]

$$\mathcal{L}(n,l,m,p) = \int \Big[\frac{\phi_x \phi_t}{2} + \alpha \frac{(\phi_x)^{p+2}}{(p+1)(p+2)} - \beta (\phi_x)^m (\phi_{xx})^2 + \frac{\gamma}{2} (\phi_x)^n (\phi_{xx})^l (\phi_{xxx})^2 \Big], \quad (16)$$

where α, β and γ are parameters adjusting the relative strentgh of the interactions, and n, l, m, p are integers. For example, for $\gamma = 0$ one re-obtains the K(2,2) equation, and for $\gamma = m = 0, p = 1$ one obtains the KdV equation. The associated Euler-Lagrange equation in the function $\phi_x = u(x,t) \to u(x - Vt) = u(y)$, reads after one integration

$$\begin{aligned} Vu = {} & \frac{\alpha}{p+1} u^{p+1} - \beta m u^{m-1} (u_y)^2 + 2\beta (u^m u_y)_y + \frac{\gamma n}{2} u^{n-1} (u_y)^l (u_{yy})^2 \\ & - \frac{\gamma l}{2} (u^n (u_y)^{l-1} (u_{yy})^2)_y + \gamma (u^n (u_y)^l u_{yy})_{yy} + C, \end{aligned} \quad (17)$$

where C is the integration constant. By using the OSA we obtain the following important result, expressed in the second row of Table 3: The unique case when such an equation allows compact supported traveling solutions is when $m = p = n + r$, $C = 0$ and $V = V_0 A^m$. This result is in full agrement with the variational calculation in [13].

Both eqs.(13) and (17) are rather more qualitative than capable of modeling measurable phenomena. That is why we introduce now a more general model equation, in the form

$$u_t + f(u)_x + g(u)_{xx} + h(u)_{xxx} = 0, \quad (18)$$

where f, g and h are differentiable functions of the the function $u(x,t)$ itself. The OSA approach gives the equation

$$-V + f'(A) + \frac{A g''(A) + g'(A)}{L} + \frac{A^2 h'''(A) + 3A h''(A) + h'(A)}{L^2} = 0. \quad (19)$$

A general analysis of eq.(19) is difficult, and the best ways are numerical investigations obtained for particular choices of the three functions. We confine ourselves here only to show that the class of solutions which have similarity properties are those for which $V = V_0 f'(A)$. In this case eq.(19) can be reduced to

$$L^2 f'(1 - V_0) + L(Ag'' + g') + A^2 h''' + 3Ah'' + h' = 0, \quad (20)$$

case which is presented in the third row of Table 3. This last relation can be used for different purposes. For example, given a certain type of dispersion and difusion (g, h fixed), we can estimate for what types of nonlinearity (f) the width L will have a given dependence with A. Or, if we know for instance $f(u) = f_0 u^{q_1}$ and $h(u) = h_0 u^{q_2}$, we can ask what type of diffusion g we need, to have constant scale (width) of the patterns (waves), no matter of the magnitude of the amplitude A. In other words, which is the compatible diffusion term, for given nonlinearity-dispersion terms, which provides fixed scale solutions. The result is obtained by integration eq.(20) with respect to $g(u)$

$$g(u) = -\frac{h_0}{L}\Big(1 + q_2 + \frac{1}{q_2 - 1}\Big)u^{q_2} - \frac{Lf_0(1 - V_0)}{q_1 - 1}u^{q_1} + C_3 \mathrm{Log} u + C_4, \tag{21}$$

where $C_{3,4}$ are constants of integration. In a similar way one can check the existence of different other configurations by solving eq.(20), or more general, eq.(19).

A last application of this method, occurs if the KdV equation has an additional term depending on the square of the curvature

$$u_t + uu_x + u_{xxx} + \epsilon(u_{xx}^2)_x = 0. \tag{22}$$

This is the case for extremely sharp surfaces (surface waves in solids or granular materials) when the hydrodynamic surface pressure cannot be linearized in curvature. Such a new term yields a new type of localized solution fulfilling the relations

$$L = \sqrt{\frac{4\epsilon A}{\pm\sqrt{1 - 8\epsilon A(A \pm V)} - 1}}.$$

If we look for a constant half-width solution (compacton of $1/L = \alpha$) we need a dependence of velocity of the form $V = (1 + \alpha^2\epsilon/8)A + 1/8\epsilon A + \alpha/4$. There are many new effects in this situation. The non-monoton dependence of the speed on A introduces again bifurcations of a unique pulse in dublets and triplets. Also, there is an upper bound for the amplitude at some critical values of the width. Pulses narrower than this critical width drop to zero. Such bumps can exist in pairs of identical amplitude at different widths. They may be related with the recent observed "oscillations" in granular materials [16, 22].

The examples presented in Tables 1-3 prove that the above method provides a reliable criterion for finding compact suported solutions. The reason this simple prescription works in so many cases follows from the advantages of wavelet analysis on localized solutions. We stress that this method has little to do with the traditional similarity (dimensional) analysis [4, 12, 13, 16, 17, 23]. In the latter case one obtains relations among powers of A, L and V, not relations with numeric coefficients like those found in our method.

3 The frame of KAK pairs

In the following we investigate the posibility of construction of a nonlinear frame (an over determined or incomplete basis) by using some compact solutions of the K(2,2) equation. The high stability against scattering of the K(2,2) compactons, or compacton generation from compact initial data, suggest they may play the role of a nonlinear local basis. We know from many numerical experiments [12, 13, 15], that any positive compact initial data decomposes into a finite series of compactons and anticompactons. This suggests that an intrinsic ingredient for a nonlinear basis could be the multiresolution structure of the solutions, similar with the structure of scaling functions in wavelet theory.

The compactons given in eqs.(5,6) have constant half-width and hence describe a unique scale, which can cover all the space by integer translations. From the point of view of multi-resolution analysis, the K(2,2) equations act like a L-band filter, allowing only a particular scale to emerge for any given set of initial condition. To each scale, from zero to infinity, we can associate a K(2,2) equation with different coefficients. However, the compacton solution is not the unique one with this property. For a given K(2,2) equation, we can thus extend the scale from L to any larger scale. These more general compact supported solutions are still $C_2(\mathbf{R})$ and are combinations of piece-wise constant and piece-wise $\cos^2$ functions. The simplest shape is given by a half-compacton prolonged with a constant level, that is a kink solution. The basis solution is a kink-antikink (KAK) compact supported combination, Fig. 1. Such kink-antikink pairs of different length, can be associated with other compactons, or KAK pairs, one on the top of the other

$$\eta_{comp+KAK}(x-Vt;\lambda)=\begin{cases} 0... \\ \frac{4V}{3}\ cos^2\left[\frac{x-Vt}{4}\right], & -2\pi\le x-Vt\le 0 \\ \frac{4V}{3}, & 0\le x-Vt\le\delta \\ \frac{4V}{3}+\frac{4}{3}(V'-2V)\ cos^2\left[\frac{x-V't}{4}\right], & \delta\le x-Vt\le\delta+4\pi \\ \frac{4V}{3}, & \delta+4\pi\le x-Vt\le\lambda \\ \frac{4V}{3}\ cos^2\left[\frac{x-Vt-\lambda}{4}\right], & \lambda\le x-Vt\le\lambda+2\pi \\ 0... \end{cases} \tag{23}$$

where $\delta<\lambda$ characterizes the initial position (at $t=0$) of the top compacton, with respect to the flat part of the KAK solution. The amplitude $4(V'-2V)/3$ of the compacton, and the amplitude $4V/3$ of the KAK, are related to their velocities V' and V, respectively.

The length of the flat part, λ, is arbitrary. A compound solution is not stable in time since the different elements travel with different velocities. The total height of the compacton is $4(V'-V)/3$. Since the higher the amplitude is, the faster the structure travels, the top compacton moves faster than the KAK, and at a certain moment it passes the KAK. Because the area of the solution is conserving, such a compound structure decomposes into compactons and KAK pairs. Similar and even more complicated constructions can be imagined, with indefinite number of compactons and KAK's, if one just fulfills the C_3 continuity condition for the square of the total structure. Such structures, defined at the initial moment can interpolate any function, playing a similar role with wavelets or spline bases. It has been also proved that the KAK solutions are stable, by using both a linear stability analysis and Lyapunov stability criteria.

For a given K(2,2) equation, the compacton solution, eq.(5) and in addition the family of KAK solutions, eq.(6) can be organized as a scaling functions system. They act like a low-pass filter in terms of space-time scales and give the opportunity to construct frames of functions from the wavelet model [5, 6, 14].

With the notation from Appendix 2, and from eq.(32), we have the elements of the frame

$$\eta_{k,j}(x) = \eta_{kak}(\pi(x - 2^j Vt - k), 2^j - 1)|_{t=0},$$

where $t = 0$ means that we neglect the time evolution, but the amplitude is still amplified with a factor of 2^j, in virtue of relation $\eta_{max} = 4V/3$. We can now expand any initial data for the K(2,2) equation in this frame

$$u_0(x) = \sum_k \sum_j C_{k,j} \eta_{k,j}(x), \tag{24}$$

and taking into account that

$$\eta_{k,j}\eta_{k',j'} \begin{cases} \neq 0 & k' = k \cdot 2^{j'-j}, ..., (k+1) \cdot 2^{j'-j} - 1 \\ = 0 & \text{otherwise,} \end{cases} \tag{25}$$

we can show that the square of the initial data can be linearized by

$$u^2(x) = \sum_{k,j} \sum_{j' \geq j} \sum_{k' \in I} C_{k,j} C_{k',j'}$$

$$\times \left(\sum_{i_1=0}^{1} \sum_{i_2=0}^{1} ... \sum_{i_{j'-j}=0}^{1} \eta_{\sigma(i_1, i_2, ..., i_{j'-j}), j'} \right) \eta_{k',j'}, \tag{26}$$

where I is the range of k' described in the first line of eq. (25), and

$$\sigma(i_1, i_2, ..., i_{j'-j}) = \sum_{l=1}^{j'-j} i_l 2^{j'-j'l+\frac{(j'-j)(j'-j+1)-l(l+1)}{2}} + k2^{(j'-j)j+\frac{(j'-j)(j'-j+1)}{2}}. \tag{27}$$

From this relation and from eq.(25) we notice that in eq.(27) the unique nonzero terms are those for which $\sigma(i_1, i_2, ..., i_{j'-j}) = k'$ with $k' \in I$. Hence the initial data is expanded in different scales at different translations. The translations are mutually orthogonal so they do not give a contribution to the square. In the multiplication of two different scales in the expression of the square, we reduce the wider scale in terms of linear combination of the narrower ones, by using the two-scale equation, eq.(31) in Appendix 2. All the nonzero terms in this product are of the order $(2^{-j}-1)/(2^{-j'}-1) \simeq 2^{j'-j}$. This number, given by the number of solutions of the equations $\sigma(i_1, i_2, ..., i_{j'-j}) = k'$, with $k' \in I$, is much smaller than the initial number of terms, where from the advantage of the frame. This is the advantage of treating nonlinear problems with a basis that has a scale criterion.

Another application of the KAK basis occurs when one needs to understand the dynamics of the initial data for the K(2,2) equation. Many numerical simulations show that in the case when the width of the initial data is larger that $L_{compacton}$, the initial shape decomposes into a finite number of compactons having the same width and different amplitudes [4, 13, 17, 18]. If we take such an initial pulse, Fig. 5a, and we want to see its time evolution untill the break up process, we have first to expand it into the KAK frame, Fig. 5a. Then we just let the system develope and the different KAK's move with their correspon velocity, eq.(23), providing the new shapes, Fig. 5b and 5c, at different moments. We notice a good agreement between these theoretical calculation and the above quoted numerical experiments. At this stage we can not yet predict the way that the KAK breaks back into compactons. However, we notice that the area of such structures in the phase space is the Poincare invariant and should give a hint towards the breaking process. This invariant is nothing that the u^3 (denoted D_2 invariant in [4]) invariant of the K(2,2) equation.

4 Comments and conclusions

First, we make a general statement concerning the compact solutions of one-dimensional NPDE. A one-dimensional dynamical model is described by a general NPDE equation

$\partial_t u = \mathcal{O}(x, \partial_x)u$ where $\mathcal{O}$ is a nonlinear differential operator. By taking into account *only* traveling solutions, this NPDE reduces to a nonlinear ordinary differential equation in the coordinate $y = x - Vt$ for an arbitrary velocity V. If $u(y)$ is a compact supported solution it follows that it is not unique with respect to fixed initial compact data. Indeed, if we choose the initial data such that the function and its derivatives (up to the requested order) are zero in the neighborhood of a certain point y_0 of the y axis, these conditions can be fulfilled by any translated version of a compact supported particular solution, placed everywhere on the axis outside this neighborhood. Consequently, for such initial data, the solution is not unique. This result shows that the compact supported property of the initial data and of the solution implies its non-uniqueness.

Since we can transform the NODE into a nonlinear differential system of order one

$$\vec{U}_y = \vec{F}(y, \vec{U}), \quad \vec{U} = (u, u_y, ...), \tag{28}$$

we can apply the fundamental theorem of existence and uniqueness to solutions of eq.(28), for given initial data $\vec{U}(y_0) = \vec{U}_0$. If the function $\vec{F}$ in eq.(28) fulfills the Lipschitz condition (its relative variation is bounded) than, for any initial condition, the solution is unique [24]. Since any linear function is analytic and hence Lipschitz, we conclude that only nonlinear functions $\vec{F}$ allow the existence of compact supported solutions. Thus, a compact soliton implies non-uniqueness in the underlying NPDE, which implies non-Lipschitzian structure of the NPDE and hence the existence of nonlinear terms.

In this paper we introduce new physical applications for wavelets, that is the study of localized solutions of nonlinear partial differential equations. The existence of compactons underlines a common feature of NPDE, discrete wavelets, and also finite differences equations. We propose a new scale approach for the similarity analysis and clasiffication of soliton solutions, without the need of solving the corresponding NPDE. Also, we proved that starting from any unique soliton solution of a NPDE, we can construct a frame of solutions organized under a multiresolution criterium. This approach provides the possibility of constructing a nonlinear basis for NPDE. We show that frames of self-similar functions are related with solitons with compact support. In addition, we notice the evidence that compactons fulfil both characteristics of solitons and wavelets, suggesting possible new applications. Such a unifying direction between nonlinearity and self-similarity, can bring new applications of wavelets in cluster formation, at any scale, from supernovae through fluid dynamics to atomic and nuclear systems. The scale approach can be applied with success to the physics of droplets, bubbles, patterns, fragmentation, fission and fusion.

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5 Appendices

5.1 Appendix 1

We base our proof on the similar proposition in [25], excepting that here we use Gaussian filtering instead of Morlet. We start form the the discrete wavelet expansion of the signal $u(s)$ given in eq.(2)

$$u(s) = \sum_j \sum_k C_{j,k} \Psi(2^j s - k) = \sum_{j,k} C_{j,k} \Psi_{j,k}(s),$$

in terms of integer translations (k) and dyadic dilations (2^j) of the Ψ wavelet. In order to reduce the number of scales needed, the range of the summation should be choosen as a function of s. We use in the following the asymptotic formula describing the pointwise behavior of the Gaussian wavelet series around a point s_0 of interest [26]. For a chosen s_0 and scale j, there is only one k and $|\epsilon| \leq 1$ such that the support of the corresponding $\Psi_{j,k}$ contains this point, $k = 2^j s_0 + \epsilon$. We can express the solution and its derivatives in a neighborhood of this point

$$u(s_0) \approx \Psi(-\epsilon) \sum_j C_{j,2^j s_0 + \epsilon} \equiv \sum_j u_j(s_0),$$

$$u_x(s_0) \approx -i\Psi(-\epsilon) \sum_j 2^j C_{j,2^j s_0 + \epsilon} = -i \sum_j 2^j u_j(s_0), \tag{29}$$

for ϵ being chosen enough large compared to one, and where $u_j(s_0) = \Psi(-\epsilon) C_{j,2^j s_0 + \epsilon} \approx \Psi(0) C_{j,2^j s_0} ...$ Since the coefficient $1/2^j$ represents the scale for each $\Psi_{j,k}$ Gaussian wavelet, we can define it as a characteristic half-width L_j. Also, we finally have for the n-th order derivative in s_0

$$u_{xx... \, x}(s_0) \approx \sum_j \frac{u_j(s_0)}{L_j^n}. \tag{30}$$

Eq.(30) is the multi-scale generalization of the simpler formula ii in eq(1). With eq.(30) in hand we can investigate the structure of hypothetic soliton solutions of NPDE, by choosing

s_0 in the neighborhood of the maximum value of the solution, $u(s_0) = A$. Around this maximum, such solutions can be described very well by a unique scale L, and hence the solution and its derivatives can be approximated with the corresponding dominant term, by the substitutions in eq.(1).

5.2 Appendix 2

For the sake of simplicity we will renormalize the coefficients of the K(2,2) equation such that the support of the simple compacton is one. That is, we take $\eta_c(x,t) = \eta_{kak}(\pi(x - Vt), 0)$ on the interval $|x - Vt|$ in $[-1/2, 1/2]$. We construct a multiresolution approximation of $L^2(\mathbf{R})$, that is an increasing sequence of closed subspaces V_j, $j \in \mathbf{Z}$, of $L^2(\mathbf{R})$ with the following properties [6, 14]

1. The V_j subspaces are all disjoint and their union is dense in $L^2(\mathbf{R})$.
2. For any function $f \in L^2(\mathbf{R})$ and for any integer j we have $f(x) \in V_j$ if and only if $D^{-1} f(x) \in V_{j-1}$ where D^{-1} is an operator that will be defined later.
3. For any function $f \in L^2(\mathbf{R})$ and for any integer k, we have $f(x) \in V_0$ is equivalent to $f(x - k) \in V_0$.
4. There is a function $g(x) \in V_0$ such that the sequence $g(x - k)$ with $k \in \mathbf{Z}$ is a Riesz basis of V_0.

In the case of compact solutions of K(2,2) of unit length, we chose for the space V_0 that which is generated by all translation of η_c with any integer k. The subspaces V_j for $j \geq 0$ are generated by all integer translations of the compressed version of this function, namely, by $\eta_{kak}(2^j\pi(x - Vt), 0)$. The subspaces V_j for $j \leq 0$ are generated by all integer translations of the KAK solution of length $\lambda 2^j - 1$. For example, V_{-1} is generated by $\eta_{kak}(\pi(x - 2^j Vt), 0)$. The spaces $V_j, j \geq 0$ are all solutions of K(2,2); the others are not. The function $g(x)$ is taken to be $\eta_{kak}(\pi(x - Vt), 0)$. It is not difficult to prove that these definitions fulfill restrictions one, three, and four. As for the second criterion, we define the action of the operator $D^{-1}f(x) = f(2x)$ if $f(x) \in V_j$ with a j positive integer, and $D^{-1}\eta_{kak}(\pi 2^j(x - 2^j Vt), 2^{-j} - 1) = \eta_{kak}(\pi 2^j(x - 2^{-j+1}Vt), 2^{-j+1} - 1)$ for negative j. In conclusion, we construct a frame of functions made of contractions of compactons and sequences of KAK solutions. We can write the corresponding two-scale equation which

connects the subspaces

$$\eta_{kak}(\pi(x - Vt), 1) = \eta_{kak}(\pi(x - Vt), 0) + \eta_{kak}(\pi(x - Vt - 1), 0). \tag{31}$$

We will denote generically by $\eta_{k,j}$ the elements of this frame, that is

$$\eta_{k,j}(x) = \eta_{kak}(\pi(x - 2^j Vt - k), 2^j - 1)|_{t=0}, \tag{32}$$

where $t = 0$ means that we neglect the time evolution, but the amplitude is still amplified with a factor of 2^j, in virtue of relation $\eta_{max} = 4V/3$.

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Figure Captions

- Fig. 1

  Examples of compactons, kink-antikink pairs, and mixed solutions of the K(2,2) equation, together with their velocities.

- Fig. 2

  A finite series of K(2,2) compactons emerging from initial compact data of width larger than of the compacton.

- Fig. 3

  The half-width L versus velocity V for the K(2,2) equation, for different amplitudes A. Widths larger than $L_{compacton} = 4$ behave different than narrower widths, with $L < 4$. Amplitude increases from left to right, in the range 0.01-0.85.

- Fig. 4

  The half-width L versus amplitude A, for the third ($n = 3$), forth ($n = 4$) and fifth ($n = 5$) order NLS equation, in two $V = \pm A$ cases. We notice that the higher order ($n > 3$) NLS equations have bifurcations.

- Fig. 5

  The dynamical evolution of the expansion in the KAK basis of a finite initial pulse of width L much larger than a compacton, drawn for three moments if time.

Table 1: Traveling localized solutions for nonlinear dispersive equations.

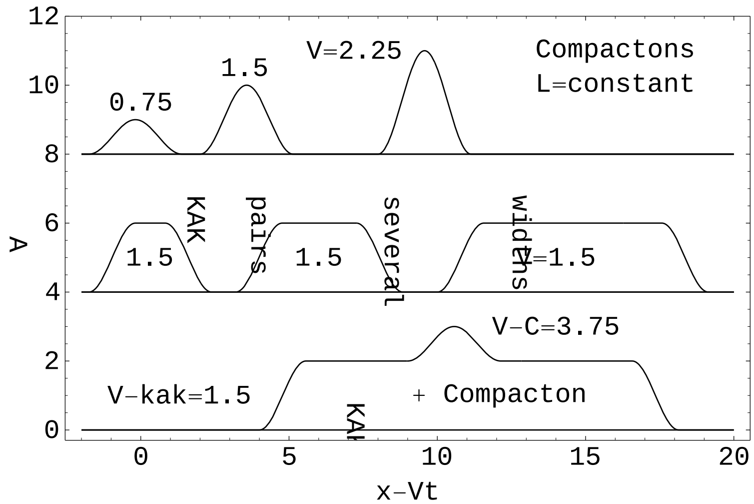
NPDE	Analytic solution and the relations among parameters	OSA approach
$u_t + 6uu_x + u_{xxx} = 0$	$A\ \mathrm{sech}^2 \frac{x-Vt}{L};\quad L = \sqrt{2/A},$ $V = 2A$	$L = \|V \pm 6A\|^{-1/2}$ If $V \sim A$, $L \sim A^{-1/2}$
$u_t + u^2 u_x + u_{xxx} = 0$	$A\ \mathrm{sech}\ \frac{x-Vt}{L};\quad L = 1/A,$ $A = \sqrt{V}$	$L = \|V \pm 6A^2\|^{-1/2}$ If $V \sim A^2, L \sim A^{-1}$
$u_t + (u^2)_x + (u^2)_{xxx} = 0$	$A\cos^2 \frac{x-Vt}{L},\quad \text{if} \quad \|(x - Vt)/4\| \leq \pi/2;$ L=4	$L = \left(\frac{8A}{\|V \pm 2A\|}\right)^{1/2}$
$u_t + (u^n)_x + (u^n)_{xxx} = 0$	$\left[Acos^2\left(\frac{x-Vt}{L}\right)\right]^{\frac{1}{n-1}},\quad \text{if } \|x - Vt\| \leq \frac{2n\pi}{n-1}$ and 0 else; $L = \frac{4n}{(n-1)},\quad A = \frac{2Vn}{n+1}$	$L = \left(\frac{n(n^2+1)}{\alpha \pm n}\right)^{1/2}$ if $V = \alpha A^{n-1}$
$u_t + (u^n)_x + (u^m)_{xxx} = 0$ $n \neq m$	unknown in general	$L = \left(\frac{n(n^2+1)A^{n-1}}{V \pm mA^{m-1}}\right)^{1/2}$

Table 2: Traveling localized solutions for nonlinear diffusive equations.

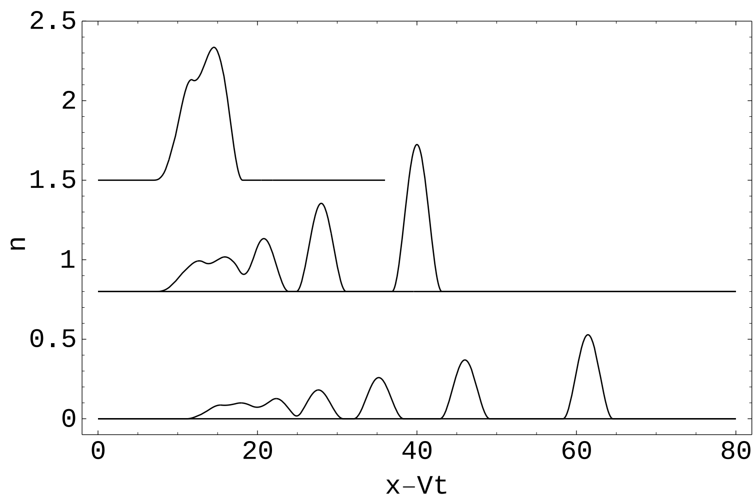
NPDE	Analytic solution and the relations among parameters	OSA approach
$u_{xx} - \frac{1}{c^2} u_{tt} = 0$	$\sum C_k e^{i(kx \pm \omega t)}$; $k^2 = \omega^2/c^2$	$V = c$ $A,\ L$ arbitrary
$u_t + uu_x - u_{xx} = 0$	$\sqrt{C - V^2} \tan(\sqrt{C - V^2} \frac{x - Vt}{2} + D)$ $+V$	$L = (A \pm V)^{-1}$ If $V \sim A$, $L \sim 1/A$
$u_t + a(u^m)_x - \mu(u^k)_{xx} + cu^\gamma$ $=0$	only particular cases known	$cA^\gamma L^2 + (V \pm amA^{m-1})L$ $\pm \mu k^2 A^{k-1} = 0$
$u_t + a(u^m)_x - \mu(u^k)_{xx} = 0$	$-\mathcal{A} z^\alpha \ {}_1F_1(\alpha, \alpha + 1, z) = x + x_0$	$L = \frac{\mu k^2}{am - \alpha} A^{k-m}$, if $V = \alpha A^{m-1}$
$u_{xt} - \sin u = 0$	$A\ \tan^{-1} \gamma\ e^{\frac{x - Vt}{L}}$	$\pm \frac{VA}{L^2} = sinA$ If $V = L^2, A = sinA$
$i\Psi_t + \Psi_{xx} + 2\|\Psi\|^2 \Psi = 0$	$\eta_0 e^{i(\omega t + kx)} sech[\eta_0(x - Vt)]$; $L = 1/\eta_0$	$L = \frac{\pm V \pm \sqrt{\|V^2 - 4A^2\|}}{2A^2}$ If $A \sim V$, $L = 1/A$
$i\Psi_t + \Psi_{xx} + \|\Psi\|^{n-1} \Psi = 0$	unknown in general	$L = \frac{\pm V \pm \sqrt{\|V^2 - 4A^n\|}}{2A^n}$
$i\Psi_t = -\frac{1}{2} \triangle \Psi$ $+[a\|\Psi\|^2 + V(x) - 1]\Psi$	$ip + \sqrt{v_c^2 - p^2} \times$ $\tanh[a\sqrt{v_c^2 - p^2}(x - q(t))]$	$L = (aA^2 \pm V - 1)^{-1/2}$ If $V \sim \pm 1, L \sim 1/(A\sqrt{a})$

Table 3: Traveling localized solutions for dissipative-dispersive equations.

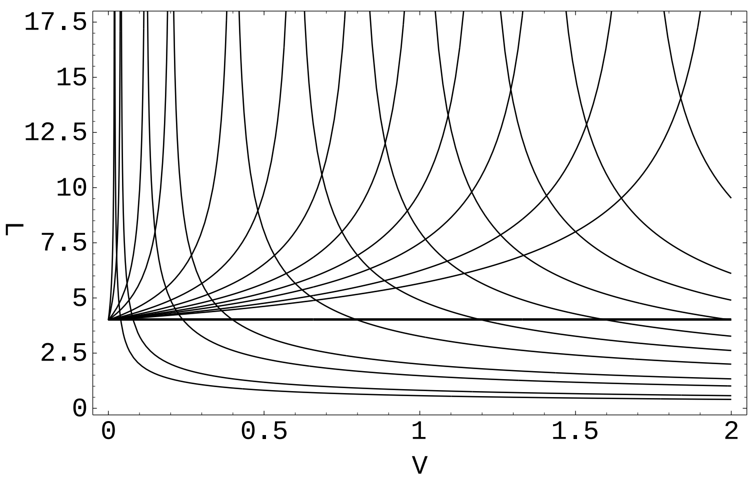
The NPDE equation	OSA approach
$u_t + a(u^m)_x + b(u^k)_{xx} + c(u^n)_{xxx} = 0;$	$L = A^{m-k} \cdot \frac{\mu k^2 \pm \sqrt{\mu^2 k^4 - 4mn^3(a-V_0)}}{2m(a-V_0)}$ if $V = mV_0 A^{k-1}$
$Vu = \frac{\alpha}{p+1} u^{p+1} - \beta m u^{m-1} (u_y)^2 + 2\beta (u^m u_y)_y$ $+\frac{\gamma n}{2} u^{n-1} (u_y)^l (u_{yy})^2 - \frac{\gamma l}{2} (u^n (u_y)^{l-1} (u_{yy})^2)_y$ $+\gamma (u^n (u_y)^l u_{yy})_{yy} + C$	$2L^{l+4}((n+l+1)V_0 - \alpha)$ $-2L^{l+2}(l+n+1)(l+n+2)\beta$ $-(l+n+1)(2+2n^2+3l+l^2+n(5+3l))\gamma$ if $C = 0$, $V = V_0 A^m$ and $m = p = n + l$
$u_t + f(u)_x + g(u)_{xx} + h(u)_{xxx} = 0$	$L = -\Big[g' + Ag'' \mp \Big((Ag'' + g')^2$ $-4f'(1-V_0)(A^2 h''' + 3Ah'' + h')\Big)^{1/2}\Big]$ $\times (2A^2 h''' + 6Ah'' + 2h')^{-1}$ if $V = V_0 f'(A)$

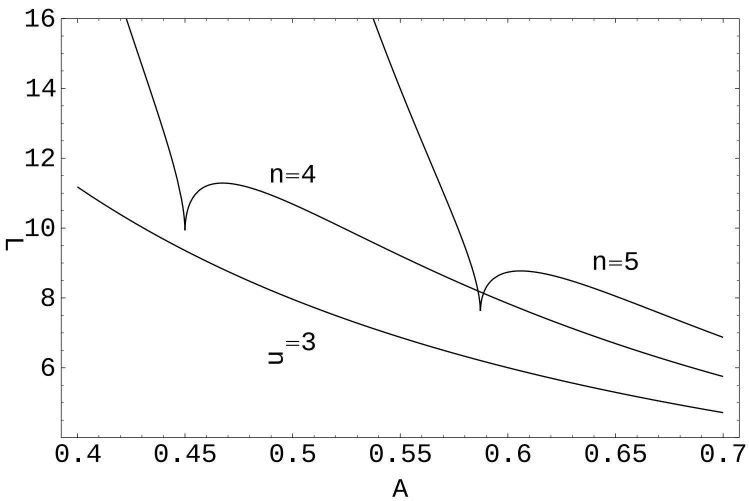


Compactons
L=constant
V=2.25
1.5
0.75
KAK
pairs
several
widths
1.5
1.5
V=1.5
V-C=3.75
V-kak=1.5
+ Compacton
KAK
A
x-Vt

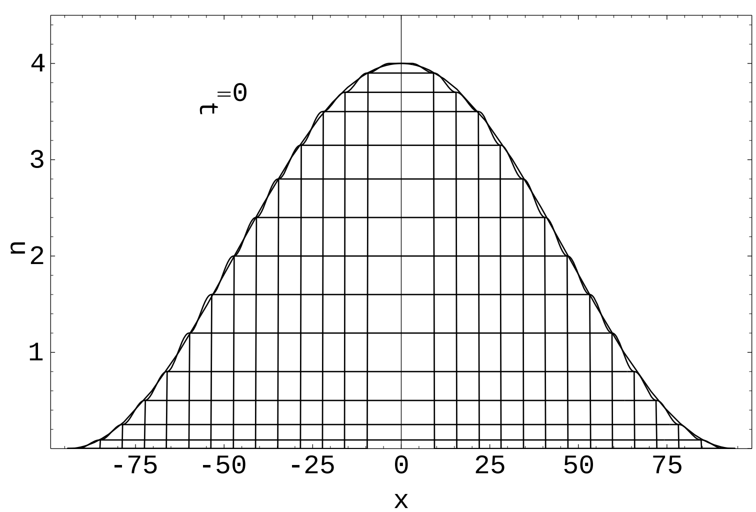


2.5
2
1.5
1
0.5
0
n
0
20
40
60
80
x-Vt

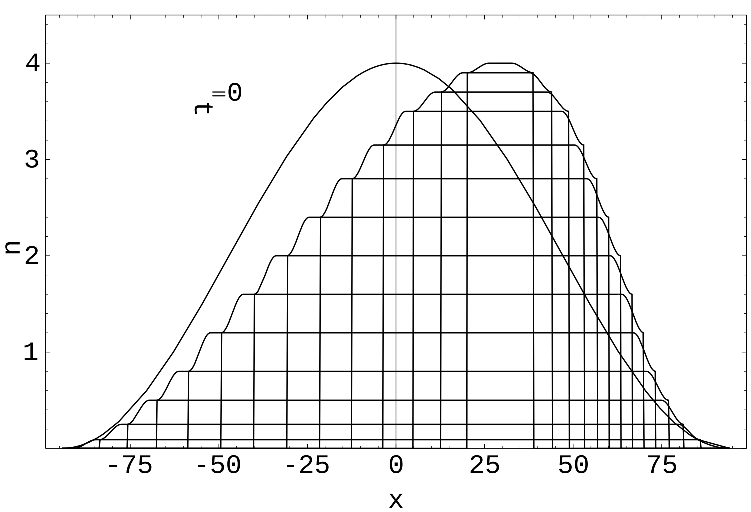

17.5
15
12.5
10
7.5
5
2.5
0
T
0
0.5
1
1.5
2
V



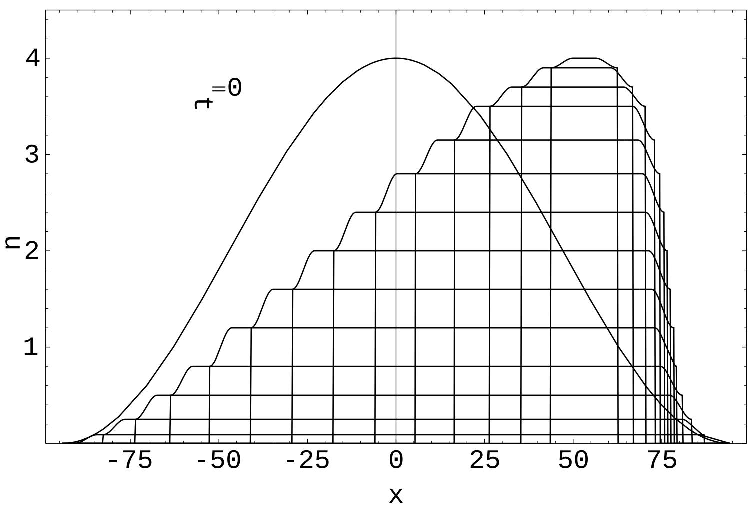
16
14
12
10
8
6
L
0.4
0.45
0.5
0.55
0.6
0.65
0.7
A
n=4
n=5
n=3



t=0
4
3
2
1
n
-75
-50
-25
0
25
50
75
x



t=0
n
4
3
2
1
-75
-50
-25
0
25
50
75
x



t=0
4
3
2
1
n
-75
-50
-25
0
25
50
75
x